An Inductive Construction for Plane Laman Graphs via Vertex Splitting

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Abstract. We prove that all planar Laman graphs (i.e. minimally generically rigid graphs with a non-crossing planar embedding) can be generated from a single edge by a sequence of vertex splits. It has been shown recently [6,12] that a graph has a pointed pseudo-triangular embedding if and only if it is a planar Laman graph. Due to this connection, our result gives a new tool for attacking problems in the area of pseudo-triangulations and related geometric objects. One advantage of vertex splitting over alternate constructions, such as edge-splitting, is that vertex splitting is geometrically more local.

We also give new inductive constructions for duals of planar Laman graphs and for planar generically rigid graphs containing a unique rigidity circuit. Our constructions can be found in $O(n^3)$ time, which matches the best running time bound that has been achieved for other inductive contructions.

1 Introduction

The characterization of graphs for rigidity circuits in the plane and isostatic graphs in the plane has received significant attention in the last few years [2, 3,8,9,14]. The special case of planar graphs, and their non-crossing realizations has been a particular focus, in part because special inductive constructions apply [3], and in part because of special geometric realizations as pseudo-triangulations and related geometric objects [6,11,12,13].

In [3] it was observed that all 3-connected planar rigidity circuits can be generated from K_4 by a sequence of vertex splits, preserving planarity, 3-connectivity and the circuit property in every intermediate step, a result that follows by duality from the construction of 3-connected planar rigidity circuits by edge splits [2]. We extend this result in two ways. We show that all planar Laman graphs (bases for the rigidity matroid) can be generated from K_2 (a single edge) by a

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sequence of vertex splits, and all planar generically rigid graphs (spanning sets of the rigidity matroid) containing a unique rigidity circuit can be generated from K_4 (a complete graph on four vertices) by a sequence of vertex splits.

One advantage of vertex splitting over alternate constructions, such as edge-splitting, is that vertex splitting is geometrically more local. With very local moves, one can better ensure the planarity of a class of realizations of the resulting graph. This feature can be applied to give an alternate proof that each planar Laman graph can be realized as a pointed pseudo-triangulation, and that each planar generically rigid graph with a unique rigidity circuit can be realized as a pseudo-triangulation with a single non-pointed vertex.

A graph $G = (V, E)$ is a *Laman graph* if $|V| \geq 2$, $|E| = 2|V| - 3$, and

$$i(X) \leq 2|X| - 3 \tag{1}$$

holds for all $X \subseteq V$ with $|X| \geq 2$, where $i(X)$ denotes the number of edges induced by X. Laman graphs, also known as isostatic or generically minimally rigid graphs, play a key role in 2-dimensional rigidity, see [5,7,8,10,14,16]. By Laman's Theorem [9] a graph embedded on a generic set of points in the plane is infinitesimally rigid if and only if it is Laman. Laman graphs correspond to bases of the 2-dimensional *rigidity matroid* [16] and occur in a number of geometric problems (e.g. unique realizability [8], straightening polygonal linkages [12], etc.)

Laman graphs have a well-known inductive construction, called *Henneberg construction*. Starting from an edge (a Laman graph on two vertices), construct a graph by adding new vertices one by one, by using one of the following two operations:

(i) add a new vertex and connect it to two distinct old vertices via two new edges (*vertex addition*)

(ii) remove an old edge, add a new vertex, and connect it to the endvertices of the removed edge and to a third old vertex which is not incident with the removed edge (*edge splitting*)

It is easy to check that a graph constructed by these operations is Laman. The more difficult part is to show that every Laman graph can be obtained this way.

Theorem 1. *[14,8]. A graph is Laman if and only if it has a Henneberg construction.*

An embedding $G(P)$ of the graph G on a set of points $P = \{p_1, ..., p_n\} \subset R^2$ is a mapping of vertices $v_i \in V$ to points $p_i \in P$ in the Euclidean plane. The edges are mapped to straight line segments. Vertex v_i of the embedding $G(P)$ is *pointed* if all its adjacent edges lie on one side of some line through p_i. An embedding $G(P)$ is *non-crossing* if no pair of segments, corresponding to independent edges of G, have a point in common, and segments corresponding to adjacent edges have only their endvertices in common. A graph G is *planar* if it has a non-crossing embedding. By a *plane* graph we mean a planar graph together with a non-crossing embedding.

A *pseudo-triangle* is a simple planar polygon with exactly three convex vertices. A *pseudo-triangulation* of a planar set of points P is a non-crossing embedded graph $G(P)$ whose outer face is convex and all interior faces are pseudo-triangles. In a *pointed pseudo-triangulation* all vertices are pointed. A *pointed-plus-one pseudo-triangulation* is a pseudo-triangulation with exactly one non-pointed vertex. The following result, due to Streinu, generated several exciting questions in computational geometry.

Theorem 2. *[12] Let G be embedded on the set $P = \{p_1, ..., p_n\}$ of points. If G is a pointed pseudo-triangulation of P then G is a planar Laman graph.*

One of the first questions has been answered by the next theorem. The proof used the following topological version of the Henneberg construction, which is easy to deduce from Theorem 1.

Lemma 1. *Every plane Laman graph has a plane Henneberg construction, where all intermediate graphs are plane, and at each step the topology is changed only on edges and faces involved in the Henneberg step. In addition, if the outer face of the plane graph is a triangle, there is a Henneberg construction starting from that triangle.*

Theorem 3. *[6] Every planar Laman graph can be embedded as a pointed pseudo-triangulation.*

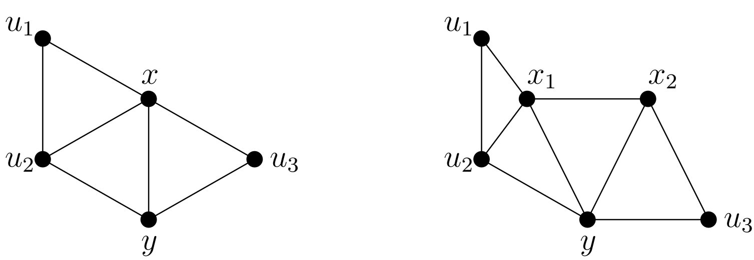

Fig. 1. The plane vertex splitting operation applied to a plane Laman graph on edge xy with partition $E_1 = \{xu_1, xu_2\}, E_2 = \{xu_3\}$.

Our main theorem is a different inductive construction for plane Laman graphs. It can also be used to prove Theorem 3 and it might be more suitable than the Henneberg construction for certain geometric problems. The construction uses (the topological version of) vertex splittings, called *plane vertex splitting*. This operation picks an edge xy in a plane graph, partitions the edges incident to x (except xy) into two consecutive sets E_1, E_2 of edges (with respect to the natural cyclic ordering given by the embedding), replaces x by two vertices x_1, x_2, attaches the edges in E_1 to x_1, attaches the edges in E_2 to x_2, and adds the edges yx_1, yx_2, x_1x_2, see Figure 1. It is easy to see that plane vertex splitting, when applied to a plane Laman graph, yields a plane Laman graph. Note that the standard version of vertex splitting (see [15,16] for its applications in rigidity theory), where the partition of the edges incident to x is arbitrary, preserves the property of being Laman, but may destroy planarity.

We shall prove that every plane Laman graph can be obtained from an edge by plane vertex splittings. To prove this we need to show that the inverse operation of plane vertex splitting can be performed on every plane Laman graph on at least three vertices in such a way that the graph remains (plane and) Laman. The inverse operation contracts an edge of a triangle face.

Let $e = uv$ be an edge of G. The operation *contracting* the edge e identifies the two end-vertices of e and deletes the resulting loop as well as one edge from each of the resulting pairs of parallel edges, if there exist any. The graph obtained from G by contracting e is denoted by G/e. We say that e is *contractible* in a Laman graph G if G/e is also Laman. Observe that by contracting an edge e the number of vertices is decreased by one, and the number of edges is decreased by the number of triangles that contain e plus one. Thus a contractible edge belongs to exactly one triangle of G.

Laman graphs in general do not necessarily contain triangles, see e.g. $K_{3,3}$. For plane Laman graphs we can use Euler's formula to deduce the following.

Lemma 2. *Every plane Laman graph $G = (V, E)$ with $|V| \geq 4$ contains at least two triangle faces (with distinct boundaries).*

It is easy to observe that an edge of a triangle face of a plane Laman graph is not necessarily contractible. In addition, a triangle face may contain no contractible edges at all. See Figure 2 and Figure 3 for examples. This is one reason why the proof of the inductive construction via vertex splitting is more difficult than that of Theorem 1, where the corresponding inverse operations of vertex addition or edge splitting can be performed at *every* vertex of degree two or three.

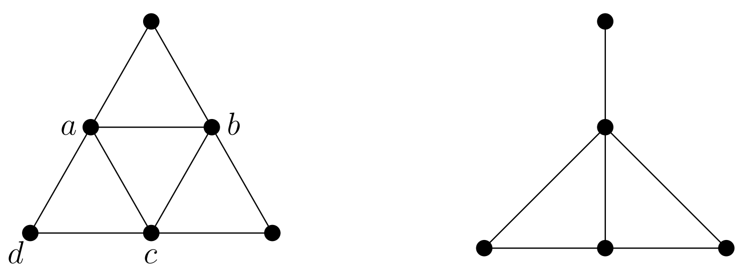

Fig. 2. A Laman graph G and a non-contractible edge ab on a triangle face abc. The graph obtained by contracting ab satisfies (1), but it has less edges than it should have. No edge on abc is contractible, but edges ad and cd are contractible in G.

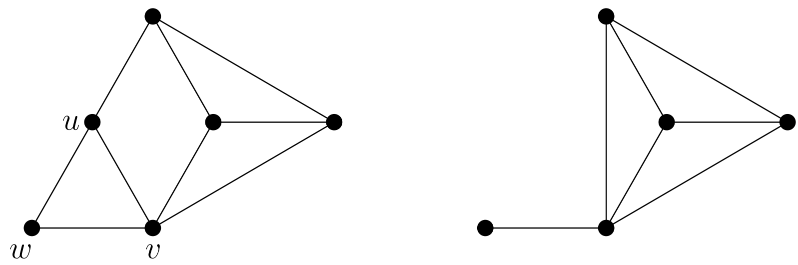

Fig. 3. A Laman graph G and a non-contractible edge uv on a triangle face uvw. The graph obtained by contracting uv has the right number of edges but it violates (1).

The paper is organised as follows. In Section 2 we prove some basic facts about Laman graphs. In Section 3 we turn to plane Laman graphs and complete the proof of our main result. Sections 4 and 5 contain some corollaries concerning pseudo-triangulations and further inductive constructions obtained by planar duality. Algorithmic issues are discussed in Section 6.

2 Preliminary Results on Laman Graphs

In this section we introduce the basic definitions and notation, and prove some preliminary lemmas on (not necessarily planar) Laman graphs.

Let $G = (V, E)$ be a graph. For a subset $X \subseteq V$ let $G[X]$ denote the subgraph of G induced by X. For a pair $X, Y \subseteq V$ let $d_G(X, Y)$ denote the number of edges from $X - Y$ to $Y - X$. We omit the subscript if the graph is clear from the context. The following equality is well-known. It is easy to check by counting the contribution of an edge to each of the two sides.

Lemma 3. *Let $G = (V, E)$ be a graph and let $X, Y \subseteq V$. Then*

$$i(X) + i(Y) + d(X, Y) = i(X \cup Y) + i(X \cap Y).$$

Let $G = (V, E)$ be a Laman graph. It is easy to see that Laman graphs are *simple* (i.e they contain no multiple edges) and 2-vertex-connected. A set $X \subseteq V$ is *critical* in G if $i(X) = 2|X| - 3$, that is, if X satisfies (1) with equality. Equivalently, X is critical if and only if $G[X]$ is Laman. Thus V as well as the set $\{u, v\}$ for all $uv \in E$ are critical. The next lemma is easy to deduce from Lemma 3.

Lemma 4. *Let $G = (V, E)$ be a Laman graph and let $X, Y \subseteq V$ be critical sets in G with $|X \cap Y| \geq 2$. Then $X \cap Y$ and $X \cup Y$ are also critical and $d(X, Y) = 0$ holds.*

Lemma 5. *Let $G = (V, E)$ be a Laman graph and let $X \subseteq V$ be a critical set. Let C be the union of some of the connected components of $G - X$. Then $X \cup C$ is critical.*

Proof. Let $C_1, C_2, ..., C_k$ be the connected components of $G - X$ and let $X_i = X \cup C_i$, for $1 \leq i \leq k$. We have $X_i \cap X_j = X$ and $d(X_i, X_j) = 0$ for all $1 \leq i < j \leq k$, and $\cup_{i=1}^k X_i = V$. Since G is Laman and X is critical, we can count as follows: $2|V|-3 = |E| = i(X_1 \cup X_2 \cup ... \cup X_k) = \sum_1^k i(X_i) - (k-1)i(X) \leq \sum_1^k (2|X_i|-3) - (k-1)(2|X|-3) = 2\sum_1^k |X_i| + 2(k-1)|X| - 3k + 3(k-1) = 2|V|-3$. Thus equality must hold everywhere, and hence each X_i is critical.

Now Lemma 4 (and the fact that $|X| \geq 2$) implies that if C is the union of some of the components of $G - X$ then $X \cup C$ is critical. □

The next lemma characterises the contractible edges in a Laman graph.

Lemma 6. *Let $G = (V, E)$ be a Laman graph and let $e = uv \in E$. Then e is contractible if and only if there is a unique triangle uvw in G containing e and there exists no critical set X in G with $u, v \in X$, $w \notin X$, and $|X| \geq 4$.*

Proof. First suppose that e is contractible. Then G/e is Laman, and, as we noted earlier, e must belong to a unique triangle uvw. For a contradiction suppose that X is a critical set with $u, v \in X$, $w \notin X$, and $|X| \geq 4$. Then e is an edge of $G[X]$ but it does not belong to any triangle in $G[X]$. Hence by contracting e we decrease the number of vertices and edges in $G[X]$ by one. This would make the vertex set of $G[X]/e$ violate (1) in G/e. Thus such a critical set cannot exist.

To see the 'if' direction suppose that there is a unique triangle uvw in G containing e and there exists no critical set X in G with $u, v \in X$, $w \notin X$, and $|X| \geq 4$. For a contradiction suppose that $G' := G/e$ is not Laman. Let v' denote the vertex of G' obtained by contracting e. Since G is Laman and e belongs to exactly one triangle in G, it follows that $|E(G')| = 2|V(G')| - 3$, so there is a set $Y \subset V(G')$ with $|Y| \geq 2$ and $i_{G'}(Y) \geq 2|Y| - 2$. Since G' is simple and uv belongs to a unique triangle in G, it follows that $V(G')$, all two-element subsets of $V(G')$, all subsets containing v' and w, as well as all subsets not containing v' satisfy (1) in G'. Thus we must have $|Y| \geq 3$, $v' \in Y$ and $w \notin Y$. Hence $X := (Y - v') \cup \{u, v\}$ is a critical set in G with $u, v \in X$, $w \notin X$, and $|X| \geq 4$, a contradiction. This completes the proof of the lemma. □

Thus two kinds of substructures can make an edge $e = uv$ of a triangle uvw non-contractible: a triangle uvw' with $w' \neq w$ and a critical set X with $u, v \in X$, $w \notin X$ and $|X| \geq 4$. Since a triangle is also critical, these substructures can be treated simultaneously. We say that a critical set $X \subset V$ is a *blocker* of edge $e = uv$ (with respect to the triangle uvw) if $u, v \in X$, $w \notin X$ and $|X| \geq 3$.

Lemma 7. *Let uvw be a triangle in a Laman graph $G = (V, E)$ and suppose that $e = uv$ is non-contractible. Then there exists a unique maximal blocker X of e with respect to uvw. Furthermore, $G - X$ has precisely one connected component.*

Proof. There is a blocker of e with respect to uvw by Lemma 6. By Lemma 4 the union of two blockers of e, with respect to uvw, is also a blocker with respect to uvw. This proves the first assertion. The second one follows from Lemma 5: let C be the union of those components of $G - X$ that do not contain w, where X is the maximal blocker of e with respect to uvw. Since $X \cup C$ is critical, and does not contain w, it is also a blocker of e with respect to uvw. By the maximality of X we must have $C = \emptyset$. Thus $G - X$ has only one component (which contains w). □

Since a blocker X is a critical set in G, $G[X]$ is also Laman.

Lemma 8. *Let $G = (V, E)$ be a Laman graph, let uvw be a triangle, and let $f = uv$ be a non-contractible edge. Let X be the maximal blocker of f with respect to uvw. If $e \neq f$ is contractible in $G[X]$ then it is contractible in G.*

Proof. Let $e = rz$. Since e is contractible in $G[X]$, there exists a unique triangle rzy in $G[X]$ which contains e. For a contradiction suppose that e is not contractible in G. Then by Lemma 6 there exists a blocker of e with respect to rzy, that is, a critical set $Z \subset V$ with $r, z \in Z$, $y \notin Z$, and $|Z| \geq 3$. Lemma 4 implies that $Z \cap X$ is critical. If $|Z \cap X| \geq 3$ then $Z \cap X$ is a blocker of e in $G[X]$, contradicting the fact that e is contractible in $G[X]$.

Thus $Z \cap X = \{r, z\}$. We claim that $w \notin Z$. To see this suppose that $w \in Z$ holds. Then $w \in Z - X$. Since $e \neq f$, and $|Z \cap X| = 2$, at least one of u, v is not in Z. But this would imply $d(X, Z) \geq 1$, contradicting Lemma 4. This proves $w \notin Z$.

Clearly, $Z - X \neq \emptyset$. Thus, since $Z \cup X$ is critical by Lemma 4, it follows that $Z \cup X$ is a blocker of f in G with respect to uvw, contradicting the maximality of X. This proves the lemma. □

3 Plane Laman Graphs

Lemma 9. *Let $G = (V, E)$ be a plane Laman graph, let uvw be a triangle face, and let $f = uv$ be a non-contractible edge. Let X be the maximal blocker of f with respect to uvw. Then all but one faces of $G[X]$ are faces of G.*

Proof. Consider the faces of $G[X]$ and the connected component C of $G - X$, which is unique by Lemma 7. Clearly, C is within one of the faces of $G[X]$. Thus all faces, except the one which has w in its interior, is a face of G, too. □

The exceptional face of $G[X]$ (which is not a face of G) is called the *special face* of $G[X]$. Since the special face has w in its interior, and uvw is a triangle face in G, it follows that the edge uv is on the boundary of the special face. If the special face of $G[X]$ is a triangle uvq, then the third vertex q of this face is called the *special vertex* of $G[X]$. If the special face of $G[X]$ is not a triangle, then X is a *nice blocker*. We say that an edge e is *face contractible* in a plane Laman graph if e is contractible and the triangle containing e (which is unique by Lemma 6) is a face in the given embedding.

We are ready to prove our main result on the existence of a face contractible edge. In fact, we shall prove a somewhat stronger result which will also be useful when we prove some extensions later.

Theorem 4. *Let $G = (V, E)$ be a plane Laman graph with $|V| \geq 4$. Then*
(i) if uvw is a triangle face, $f = uv$ is not contractible, and X is the maximal blocker of f with respect to uvw, then there is an edge in $G[X]$ which is face contractible in G,
(ii) for each vertex $r \in V$ there exist at least two face contractible edges which are not incident with r.

Proof. The proof is by induction on $|V|$. It is easy to check that the theorem holds if $|V| = 4$ (in this case G is unique and has essentially one possible planar embedding). So let us suppose that $|V| \geq 5$ and the theorem holds for graphs with less than $|V|$ vertices.

First we prove (i). Consider a triangle face uvw for which $f = uv$ is not contractible, and let X be the maximal blocker of f with respect to uvw. Since X is a critical set, the induced subgraph $G[X]$ is Laman. Together with the embedding obtained by restricting the embedding of G to the vertices and edges of its subgraph induced by X, the graph $G[X]$ is a plane Laman graph. Since $w \notin X$, $G[X]$ has less than $|V|$ vertices.

We call an edge e of $G[X]$ *proper* if $e \neq f$, e is face contractible in $G[X]$, and the triangle face of $G[X]$ containing e is a face of G as well. It follows from the definition and Lemma 8 that a proper edge e is face contractible in G as well. We shall prove (i) by showing that there is a proper edge in $G[X]$.

To this end first suppose that $|X| = 3$. Then $G[X]$ is a triangle, and each of its edges is contractible in $G[X]$. By Lemma 9 one of the two faces of $G[X]$ is a face of G as well. Thus each of the two edges of $G[X]$ which are different from f, is proper.

Next suppose that $|X| \geq 4$. By the induction hypothesis (ii) holds for $G[X]$ by choosing $r = u$. Thus there exist two face contractible edges e', e'' in $G[X]$ which are not incident with u (and hence e' and e'' must be different from f). If X is a nice blocker then the triangle face containing e' (or e'') in $G[X]$ is a face of G as well, by Lemma 9. Thus e' (or e'') is proper.

If X is not a nice blocker then it has a special triangle face uvq, which is not a face of G, and each of the other faces of $G[X]$ is a face of G by Lemma 9. Since e' and e'' are distinct edges which are not incident with u, at least one of them, say e', is not an edge of the triangle uvq. Hence the triangle face of $G[X]$ containing e' is a triangle face of G as well. Thus e' is proper. This completes the proof of (i).

It remains to prove (ii). To this end let us fix a vertex $r \in V$. We have two cases to consider.

Case 1. There exists a triangle face uvw in G with $r \notin \{u, v, w\}$.

If at least two edges on the triangle face uvw are face contractible then we are done. Otherwise we have blockers for two or three edges of uvw.

If none of the edges of the triangle uvw is contractible then there exist maximal blockers X, Y, Z for the edges vw, uw, and uv (with respect to u, v, and w), respectively. By Lemma 4 we must have $X \cap Y = \{w\}$, $X \cap Z = \{v\}$, and $Y \cap Z = \{u\}$ (since the sets are critical and $d(Y, Z), d(X, Y), d(X, Z) \geq 1$ by the existence of the edges of the triangle uvw). By our assumption r is not a vertex of the triangle uvw. Thus r is contained by at most one of the sets X, Y, Z. Without loss of generality, suppose that $r \notin X \cup Y$. By (i) each of the subgraphs $G[X], G[Y]$ contains an edge which is face contractible in G. These edges are distinct and avoid r. Thus G has two face contractible edges not containing r, as required.

Now suppose that uv is contractible but vw and uw are not contractible. Then we have maximal blockers X, Y for the edges vw, uw, respectively. As above, we must have $X \cap Y = \{w\}$ by Lemma 4. Since $r \neq w$, we may assume, without loss of generality, that $r \notin X$. Then it follows from (i) that there is an edge f in $G[X]$ which is face contractible in G. Thus we have two edges (uv and f), not incident with r, which are face contractible in G.

Case 2. Each of the triangle faces of G contains r.

Consider a triangle face ruv of G. Then uv is face contractible, for otherwise (i) would imply that there is a face contractible edge in $G[X]$, (in particular, there is a triangle face of G which does not contain r), a contradiction. Since G has at least two triangle faces by Lemma 2, it follows that G has at least two face contractible edges avoiding r. This proves the theorem. □

Theorem 4 implies that a plane Laman graph on at least four vertices has a face contractible edge. A plane Laman graph on three vertices is a triangle, and each of its edges in face contractible. Note that after the contraction of a face contractible edge the planar embedding of the resulting graph can be obtained by a simple local modification.

Since contracting an edge of a triangle face is the inverse operation of plane vertex splitting, the proof of the following theorem by induction is straightfoward.

Theorem 5. *A graph is a plane Laman graph if and only if it can be obtained from an edge by plane vertex splitting operations.*

Theorem 4 implies that if G has at least four vertices then there is a face contractible edge avoiding any given triangle face. Thus any triangle face can be chosen as the starting configuration in Theorem 5.

We say that $G = (V, E)$ is a *rigidity circuit* if $G - e$ is Laman for every edge e of G. We call it a *Laman-plus-one graph* if $G - e$ is Laman for some edge e of G. From the matroid viewpoint it is clear that Laman-plus-one graphs are those rigid graphs (i.e. spanning sets of the rigidity matroid) which contain a unique rigidity circuit. In particular, rigidity circuits are Laman-plus-one graphs.

By using similar techniques we can prove the following. The proof is omitted from this version.

Theorem 6. *A graph is a plane Laman-plus-one graph if and only if it can be obtained from a K_4 by plane vertex splitting operations.*

Remark A natural question is whether a 3-connected plane Laman graph has a face contractible edge whose contraction preserves 3-connectivity as well (call such an edge *strongly face contractible*). The answer is no: let G be a 3-connected plane Laman graph and let G' be obtained from G by inserting a new triangle face $a'b'c'$ and adding the edges aa', bb', cc' (operation *triangle insertion*), for each of its triangle faces abc. Then G' is a 3-connected plane Laman graph with no strongly face contractible edges. It is an open question whether there exist good local reduction steps which could lead to an inductive construction in the 3-connected case, such that all intermediate graphs are also 3-connected.

4 Pseudo-Triangulations

Theorems 5 and 6 can be used to deduce a number of known results on pseudo-triangulations. One can give new proofs for Theorem 3 and for the following similar result (which was also proved in [6]): every planar Laman-plus-one graph can be embedded as a pointed-plus-one pseudo-triangulation. The proofs are omitted. They are similar to the proofs in [6] but the lengthy case analyses can be shortened, due to the fact that plane vertex splitting is more local.

Theorems 5 and 6 can also be used to prove the main results of [6] on *combinatorial pseudo-triangulations* of plane Laman and Laman-plus-one graphs. See [6] for more details on this combinatorial setting.

5 Planar Duals

Let $G = (V, E)$ be a graph. We say that G is *co-Laman* if $|E| = 2|V| - 1$ and $i(X) \leq 2|X| - 2$ for all proper subsets $X \subset V$. Note that co-Laman graphs may contain multiple edges. Let $M(G)$ denote the circuit matroid of G and let $M^*(G)$ denote its dual.

Theorem 7. *Let G and H be a pair of planar graphs with $M(G) \cong M^*(H)$. Then G is Laman if and only if H is co-Laman.*

Proof. (Sketch) Simple matroid facts and Nash-Williams' characterisation of graphs which can be partitioned into two spanning trees imply that G is Laman $\Leftrightarrow$ $M(G/e)$ is the disjoint union of two bases (spanning trees) for all $e \in E(G)$ $\Leftrightarrow$ $M(H) - e$ is the disjoint union of two bases (spanning trees) for all $e \in E(H)$ $\Leftrightarrow$ H is co-Laman. □

Recall the second Henneberg operation (edge splitting) from Section 1, which adds a new vertex and three new edges. If the third edge incident to the new vertex may also be connected to the endvertices of the removed edge (i.e. it may be parallel to one of the other new edges), we say that we perform a *weak edge splitting* operation. The *plane weak edge splitting* operation is the topological version of weak edge splitting (c.f. Lemma 1). It is not difficult to see that the dual of plane vertex splitting is plane weak edge splitting. Thus Theorem 5 and Theorem 7 imply the following inductive construction via planar duality.

Theorem 8. *A plane graph is co-Laman if and only if it can be obtained from a loop by plane weak edge splittings.*

6 Algorithms

In this section we describe the main ideas of an algorithm which can identify a face contractible edge in a plane Laman graph G in $O(n^2)$ time, where n is the number of vertices of G. It uses a subroutine to test whether an edge $e = uv$ of a triangle face is contractible (and to find its maximal blocker, if it is non-contractible), which is based on the following lemma. The maximal rigid subgraphs of a graph are called the *rigid components*, see [5,8] for more details.

Lemma 10. *Let G be a plane Laman graph and let $e = uv$ be an edge of a triangle face uvw of G. Then either e is contractible, or the rigid component C of $G - w$ containing e has at least three vertices. In the latter case the vertex set of C is the maximal blocker of e with respect to uvw in G.*

There exist algorithms for computing the rigid components of a graph in $O(n^2)$ time, see e.g. [1,4,7].

The algorithm identifies a decreasing sequence $V = X_0, X_1, ..., X_t$ of subsets of V such that X_{i+1} is a blocker of some triangle edge of $G[X_i]$ for $0 \leq i \leq t-1$ and terminates with a contractible edge of $G[X_t]$ which is also contractible in G. First it picks an arbitrary triangle face uvw of G and tests whether any of its edges is contractible (and computes the blockers, if they exist). If yes, it terminates with the desired contractible edge. If not, it picks the smallest blocker B found in this first iteration (which belongs to edge $f = uv$, say), puts $X_1 = B$, and continues the search for a proper edge in $G[X_1]$. To this end it picks a triangle in $G[X_1]$ (which is different from the special face of $G[X_1]$, if X_1 is not a nice blocker) and tests whether any of its edges which are different from f (and which are not on the special face of $G[X_1]$, if X_1 is not a nice blocker) is contractible in $G[X_1]$. If yes, it terminates with that edge. Otherwise it picks the smallest blocker B' found in this iteration among those blockers which do not contain f (and which do not contain edges of the special face of $G[X_1]$, if X_1 is not a nice blocker), puts $X_2 = B'$, and iterates. The following lemma, which can be proved by using Lemma 4, shows that there are always at least two blockers to choose from.

Lemma 11. *(i) Let f be an edge and let uvw be a triangle face whose edges (except f, if f is on the triangle) are non-contractible. Then at least two blockers of the edges of uvw do not contain f.*
(ii) Let xyz and uvw be two distinct triangle faces such that the edges of uvw (except those edges which are common with xyz) are non-contractible. Then at least two blockers of the edges of uvw do not contain edges of xyz.

It follows from Lemma 8, Lemma 9, and Lemma 11 that the algorithm will eventually find a proper edge e in some subgraph $G[X_t]$, and that this edge will be face contractible in G as well. The fact that the algorithm picks the smallest blocker from at least two candidates (and that the blockers have exactly one vertex in common by Lemma 4) implies that the size of X_i will be essentially halved after every iteration. From this it is not difficult to see that the total running time is also $O(n^2)$.

Theorem 9. *A face contractible edge of a plane Laman graph can be found in $O(n^2)$ time.*

Theorem 9 implies that an inductive construction via vertex splitting can be built in $O(n^3)$ time. This matches the best time bound for finding a Henneberg construction, see e.g. [6].

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